ROLE OF POWER SERIES SPACES IN THE STRUCTURE THEORY OF NUCLEAR FRECHET SPACES

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Most of the locally convex spaces appearing in the theory of distributions, as well as spaces of analytic functions of several variables, are nuclear. Many of the important examples of these spaces are either Fréchet or the dual of Fréchet spaces or they can be represented as inductive limits of Fréchet spaces or their duals. The structure theory of nuclear Fréchet spaces has captured the attention of analysts from the time of the introduction of the concept of nuclearity by A. Grothendieck. An early theorem of Dynin and Mitiagin states that if a nuclear Fréchet space has a Schauder basis, then it is canonically isomorphic to a nuclear Köthe space. Although not every nuclear Fréchet space has a Schauder basis by a result of Mitiagin and Zobin, there are many concrete examples which do. In these examples the basis is usually constructed by a Taylor expansion. Therefore it is natural to try to understand the structure of nuclear Fréchet spaces in terms of Köthe spaces. Of course among Köthe spaces, power series spaces have a prominent place.

1. Diametral Dimension

Let A and B be two absolutely convex subsets of a locally convex space E such that $A \subset \rho B$ for some $\rho > 0$. We define the n-th *Kolmogorov diameter* of A with respect to B with

$$d_n(A, B) = \inf \inf\{d > 0 : A \subset dB + L\}$$

where the second infimum is taken over all subspaces L of E with dimension not exceeding n. The n-th *Gelfand number* is similarly defined as

$$g_n(A, B) = \inf \inf\{d > 0 : A \cap M \subset dB\}$$

where the second infimum is taken over all closed subspaces M of E with codimension not exceeding n.

Let $\mathcal{U}(E)$ be a base of neighborhoods of E consisting of absolutely convex and closed subsets and $\mathcal{B}(E)$ all absolutely convex, closed and bounded subsets of E.

We define the *diametral dimension* $\Delta(E)$ of E as the set of all real sequences (ξ_n) such that for every $\mathcal{U} \in \mathcal{U}(E)$ there is a $V \in \mathcal{U}(E)$ with $\lim \xi_n d_n(V, U) = 0$. Similarly, $\Delta_B(E)$ is the set of all real sequences (ξ_n) such that $\lim \xi_n d_n(B, U) = 0$ for every $B \in \mathcal{B}(E)$ and $\mathcal{U} \in \mathcal{U}(E)$.

If we replace the Kolmogorov diameters with Gelfand numbers in the above, what we have will be denoted by $\Gamma(E)$ and $\Gamma_B(E)$ respectively.

Remarks.

1. Δ, Δ_B, Γ and Γ_B are independent of our choice of the base $\mathcal{U}(E)$ and they are isomorphic invariants. That is, if two locally convex spaces E and F are isomorphic then, $\Delta(E) = \Delta(F),\ \Delta_B(E) = \Delta_B(F), \Gamma(E) = \Gamma(F)$ and $\Gamma_B(E) = \Gamma_B(F)$.
2. We have $c_0 \subset \Delta(E) \subset \Delta_B(E)$ in general. For any infinite dimensional normed space E, it is easy to see $c_0 = \Delta(E) = \Delta_B(E)$. Therefore for normed spaces of infinite dimension, these invariants are useless.
3. Let $\omega = R^N$ denote the space of all real sequences. Then we have $\Delta(\omega) = \Delta(R^k) = \omega$ for any integer $k \geq 1$.
4. Let U be the closed unit ball of a normed space E and $A \in \mathcal{B}(E)$. A is precompact if and only if $\lim d_n(A, U) = 0$. From this we can obtain that every bounded subset of a locally convex space E is precompact if and only if $\ell_\infty \subset \Delta_B(E)$.
5. Similarly, a locally convex space E is a Schwartz space if and only if $\ell_\infty \subset \Delta(E)$.
6. Let F be a FM-space which is not a Schwartz space ([20]; §30). Then $\Delta(F) = c_0$ but $\ell_\infty \subset \Delta_B(F)$. Hence even for Fréchet spaces Δ and A_B do not coincide.

In the preceeding remarks, we can replace the Kolmogorov diameters with the Gelfand numbers or Δ with Γ, Δ_B with Γ_B and obtain the same results. All of these can be found for example in [35]. For a general references to Fréchet spaces see Jarchow [18] Meise and Vogt [25] Köthe [20]

2. Köthe Spaces

Let A be a set of non-negative sequences with the following properties:

1. $\forall i\ \ \exists a \in A$ with $a_i > 0$
2. $\forall a, b \in A\ \ \exists c \in A$ with $\max\{a_n, b_n\} \leq c_n$ for all $n \in N$

We denote by $\lambda(A)$ the set of all scalar sequences $\xi = (\xi_n)$ such that

$$p_a(\xi) = \sum |\xi_n| a_n < \infty$$

for all $a \in A$. It is easily seen that $\lambda(A)$ is a locally convex space when equipped with the semi-norms $p_a(\cdot), a \in A$.

Suppose each $a \in A$ satisfies $0 < a_1 \leq a_2 \leq \cdots$ and $\forall\ a \in A\ \ \exists b \in A$ with $a_n^2 = 0(b_n)$. We call the sequence space $\lambda(A)$ is this case a G_∞*-space* ([35]) and we have

$$\Delta(\lambda(A)) = \{\xi : \exists a \in A \ \text{ with } \xi_n = 0(a_n)\} = \lambda(A)'.$$

The notation $a_n = 0(b_n)$ simply means that there is some $\rho > 0$ with $a_n \leq \rho b_n,\ \forall\ n \in N$. Somewhat dual to the concept of a G_∞-space is G_1-spaces, which is defined by:

$$\forall\ a \in A \quad \text{we have} \quad a_n \geq a_{n+1} > 0 \text{ and}$$
$$\forall\ a \in A\ \ \exists b \in A \quad \text{with} \quad a_n = 0(b_n^2).$$

In this case we have

$$\Delta(\lambda(A)) = \lambda(A).$$

G_∞ and G_1-spaces are called *smooth sequence spaces* of infinite or finite type respectively. For further properties of these classes of sequence spaces we refer to [35], [36] and [37]. Let $0 \leq a_n^k \leq a_n^{k+1}\ \forall\ k, n \in N$ and for each $i \in N$ we assume $a_i^k > 0$ for some $k \in N$. Then the sequence space $\lambda(A)$ is a Fréchet space and it will be called a *Köthe space* in that follows. Let $0 < \alpha_1 \leq \alpha_2 \cdots$ with $\lim \alpha_n = \infty$. If

$$P = \{(e^{k\alpha_n}) : k, n \in N\}$$

the Köthe space $\lambda(P)$ will be called a *power series space of infinite-type* and it will be denoted by $\Lambda_\infty(\alpha)$.

On the other hand, if we let $Q = \{(e^{r_k\alpha_n}) : k, n \in N\}$ where $r_1 \leq \ldots \leq r_k < 0,\ \lim r_k = 0$, the Köthe space $\lambda(Q)$ will be called a *power series space of finite-type* and denoted by $\Lambda_1(\alpha)$.

$\Lambda_\infty(\alpha)$ is an example of a G_∞-space and $\Lambda_1(\alpha)$ an example of a G_1-space. From comparison of diametral dimensions, we can easily observe that $\Lambda_\infty(\alpha)$ is isomorphic to $\Lambda_\infty(\beta)$ if and only if $\alpha_n = 0(\beta_n)$ and $\beta_n = 0(\alpha_n)$. Same result is true for power series spaces of finite type. On the other hand, these two classes are essentially different, since we know that each continuous linear map from $\Lambda_1(\alpha)$ into $\Lambda_\infty(\beta)$ maps a neighborhood onto a precompact subset ([49], [13]). Such pairs of Fréchet spaces was characterized by Vogt [43]. Hence no infinite dimensional quotient space of $\Lambda_1(\alpha)$ is isomorphic to a subspace of $\Lambda_\infty(\beta)$. However, $\Lambda_\infty(\beta)$ can be isomorphic to a subspace of $\Lambda_1(\beta)$ ([14], [29]).

For two exponent sequences α and β let γ be the increasing rearrangement of the sequence $(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots)$. The direct sum $\Lambda_\infty(\alpha) \oplus \Lambda_\infty(\beta)$ is isomorphic to $\Lambda_\infty(\gamma)$. The same is true for finite-type power series spaces. In particular, $\Lambda_\infty(\alpha) \oplus \Lambda_\infty(\alpha)$ is isomorphic to $\Lambda_\infty(\alpha)$ if and only if $\alpha_{2n} = 0(\alpha_n)$. In this case we say α is *stable.* We have $\Lambda_\infty(\alpha) \oplus R^k \simeq \Lambda_\infty(\alpha)$, for $k \geq 1$, if and only if $\alpha_{n+1} = 0(\alpha_n)$. In this case we say, α is *shift-stable.* In the above, we can take finite type power series spaces instead of the infinite type and get the same results. More generally, for a G_∞-space $\lambda(P)$ we have that $\lambda(P) \oplus \lambda(P) \simeq \lambda(P)$ if and only if $\forall\ p \in P\ \ \exists\ \ p' \in P$ with $p_{2n} = 0(p'_n)$. Similarly for a G_1-space we have $\lambda(Q) \simeq \lambda(Q) \oplus \lambda(Q)$ if and only if $\forall\ q \in Q\ \ \exists\ \ q' \in Q$ with $q_n = 0(q'_{2n})$. All of these follow by diametral dimension arguments ([35]).

3. Nuclear Spaces and Diametral Dimension

Nuclear locally convex spaces were first defined and explored systematically by Grothendieck in his thesis [17]. His definition is in terms of topological tensor products and it is as follows: A locally convex space E is called *nuclear* if the completion of the π-tensor product $E\widetilde{\otimes}_\pi F$ is isomorphic to the completion of the ϵ-tensor product $E\widetilde{\otimes}_\epsilon F$ for every locally convex space F. Even a superficial glance to this definition will show how difficult it is to check whether a given locally convex space is nuclear or not. Later A. Pietsch reformulated the definition of nuclearity in terms of summable families, absolutely summing maps, nuclear, quasi-nuclear and Hilbert-Schmidt maps and thus the theory of nuclear locally convex spaces became much more accesible. The original version of Pietsch's book [32] was published in German in 1965, where several equivalent definitions of nuclearity are stated starting at p. 62. His book also contains the following characterization in terms of diametral dimension which is due to Dynin and Mitiagin [16]. (cf. [26]).

Theorem 3.1. *The following conditions are equivalent for a locally convex space E.*

(i) *E is nuclear*
(ii) *$(n^k) \in \Delta(E)$ for some $k \geq 1$*
(iii) *$(n^k) \in \Delta(E)$ for all $k \geq 1$.*

It is easy to see that the Köthe space $\lambda(A),\ A = \{(n^k) : k = 1, 2, \ldots\}$ is in fact equal to the power series space $\Lambda_\infty(\log(n+1))$, which in turn is denoted by s. This is the space of *rapidly decreasing sequences.* Hence $s' = \lambda(A)' = \{(\xi_n) : \xi_n = O(n^k)$ for some $k \geq 1\}$. Hence we can reformulate our theorem as follows

Theorem 3.2. *E is nuclear $\Leftrightarrow s' \subset \Delta(E)$.*

Since $\ell_\infty \subset s'$, we have that every nuclear space is a Schwartz space. In particular a nuclear Fréchet space is a Montel space. Also, if E is a nuclear locally convex space, then we can find a base of neighborhoods $\mathcal{U}(E)$ such that each $U \in \mathcal{U}(E)$ has a gauge

$$p_u(x) = \inf\{d > 0 : x \in dU\}$$

is actually defined by a semi-inner product $(\ |\ )_u$

$$p_u(x) = (x|x)_u^{1/2}.$$

This yields the following result

Proposition 3.3. *If E is nuclear, then $\Delta(E) = \Gamma(E)$. If F is a subspace or a quotient space of E, then $\Delta(E) \subset \Delta(F)$.*

Let us consider now the special class of sequence spaces. Both of the following results can be derived by using diametral dimension.

Proposition 3.4. *$\lambda(A)$ is a Schwartz space $\Leftrightarrow \forall\ \ a \in A\ \ \exists\ \ b \in A$ with $a_n = o(b_n)$ $a_n = o(b_n)$, which means $a_n \leq \xi_n b_n$ for some sequence (ξ_n) which converges to zero.*

Similar to the above we give a characterization of nuclear sequence spaces, which is called the Grothendieck-Pietsch criterion.

Theorem 3.5. *$\lambda(A)$ is a nuclear space $\Leftrightarrow \forall\ a \in A\ \ \exists b \in A$ and $(\xi_n) \in \ell_1$ with $a_n \leq \xi_n b_n$.*

In the definition of $\lambda(A)$, if we replace the ℓ_1-seminorm $p_a(\cdot)$ by

$$\left(\sum_{n=1}^{\infty} |\xi_n|^p\ a_n^p\right)^{1/p} < \infty, \quad 1 \leq p < \infty$$

or

$$\sup |\xi_n| a_n < \infty, \quad (\text{ case } p = \infty)$$

we get locally convex spaces denoted by $\lambda^p(A), \quad 1 \leq p \leq \infty$, which are different in general. A rather simple, but highly useful consequence of the previous theorem states that in case of nuclearity these spaces are the same.

Corollary 3.6. *If $\lambda(A)$ is nuclear, then $\lambda^p(A) = \lambda(A)$ for any $1 \leq p \leq \infty$, and the topologies are the same.*

The converse is also true but this not very useful in applications. Finally we note that $\Lambda_\infty(\alpha)$ is nuclear $\Leftrightarrow \log(n+1) = 0(\alpha_n) \Leftrightarrow (q^{\alpha_n}) \in \ell_1$ for some $0 < q < 1$. However in case of finite type power series spaces, things are a little different: $\Lambda_1(\alpha)$ is nuclear $\Leftrightarrow (q^{\alpha_n}) \in \ell_1$ for all $0 < q < 1$. More generally, a G_∞-space $\lambda(P)$ is nuclear $\Leftrightarrow s' \subset \lambda(P)' \Leftrightarrow n = O(p_n)$ for some $(p_n) \in P$. A G_1-space $\lambda(Q)$ is nuclear $\Leftrightarrow s' \subset \lambda(Q) \Leftrightarrow Q \subset s$ ([35]).

4. Bases in Nuclear Fréchet Spaces

In this section all locally convex spaces will be metrisable and complete, that is Fréchet spaces (or F-spaces). A sequence (x_n) in an F-space E is called a *basis* if $\forall \ x \in E \ \exists$ a unique sequence (ξ_n) of scalars with

$$x = \sum_{n=1}^{\infty} \xi_n x_n.$$

In this case we can find $u_n \in E'$ such that

$$x = \sum_{n=1}^{\infty} u_n(x) x_n.$$

(u_n) is called the *sequence of coordinate functionals*. For the case of sequences spaces $\lambda(A)$, if e_n denotes that sequence which has 1 as the n-th term but all other terms are zero, (e_n) is a basis, called the *canonical basis* of $\lambda(A)$. In case of nuclearity we have the following essential and powerful result, which is called the *Dynin-Mitiagin Theorem* ([16], [26]).

Theorem 4.1. *Let $(|\cdot|_k)$ be a sequence of seminorms defining the topology of a nuclear F-space E. If (x_n) is a basis of E with coordinate functionals (u_n) and $A = \{(|x_n|_k) : k = 1, 2, \ldots\}$, then the map which sends each $x \in E$ to $(u_n(x))$ is an isomorphism of E onto $\lambda(A)$.*

A Fréchet space which has a basis is separable. Every Fréchet-Schwartz space and so every nuclear F-space is also separable. Already Grothendieck had asked whether every FN-space has a basis. In the light of the Dynin-Mitiagin theorem a positive answer to Grothendieck's question would reduce any problem about nuclear Fréchet spaces to a problem about nuclear Köthe spaces. However in 1974 Mitiagin and Zobin ([50]) constructed an example of a nuclear Fréchet space which has no basis. Subsequently Djakov and Mitiagin ([12]) gave a procedure for constructing nuclear Fréchet spaces without bases. Several authors, including Bessaga, Dubinsky and Mitiagin proved among other theorems the following result.

Proposition 4.2. *Given any nuclear F-space E. There is a subspace of E which has no basis. There is also a quotient space of E which has no basis.*

However there is another problem posed by Pelczynski [31], which in general is still open.

Problem 4.3. Does every complemented subspace of a nuclear Köthe space $\lambda(A)$ have a basis?

We recall that F is a *complemented subspace* of $\lambda(A)$ if there is a continuous projection $P : \lambda(A) \to \lambda(A)$ with $P(\lambda(A)) = F$. This is equivalent to $\lambda(A) \simeq F \oplus G$ where F and G are closed subspaces of $\lambda(A)$

In special cases Pelczynski's problem was solved positively. The earliest case was for power series spaces of finite type, due to Mitiagin and Henkin [28]. (cf. [27])

Theorem 4.4. *Every complemented subspace of a nuclear power series space of finite type has a basis.*

A rather simple consequence of this theorem is that if F is a complemented subspace of a nuclear power series space $\Lambda_1(\alpha)$, then F is isomorphic to some $\Lambda_1(\beta)$, where $\alpha_n = O(\beta_n)$. In his thesis, J. Krone [21], [22] formulated an abstract version of the so-called dead-end space method of Mitiagin-Henkin and refined their theorem in several directions. He proved, for example, that if $T : \Lambda_1(\alpha) \to \Lambda_1(\alpha)$ is a continuous linear operator, then the closure of $T(\Lambda_1(\alpha))$ has a basis. With his method it is also proved that for a nuclear power series spaces $\Lambda_\infty(\alpha)$, Pelczynski's problem has a positive answer provided $\lim(\alpha_{n+1}/\alpha_n) = \infty$. (cf. [15]). However, the following problem which is a special case of Pelczynski's problem, is still unsolved in its generality.

Problem 4.5. Does every complemented subspace of a nuclear power series space of infinite type have a basis?

5. Highlights of the Structure Theory

Throughout this section all locally convex spaces under consideration will be Fréchet spaces.

We already know (3.2. Theorem) that E is nuclear if and only if $\Delta(s) = s' \subset \Delta(E)$. Since $(\log(n+1))$ is stable and s^N is nuclear, we have $\Delta(s) = \Delta(s^N)$. So s^N is an FN-space. Grothendieck has asked the question whether any FN-space E is isomorphic to a subspace of s^N. This was answered positively by Komura and Komura [19]. Some years later Ramanujan and Terzioğlu [33] extended their result as follows:

Theorem 5.1. *Let $\Lambda_\infty(\alpha)$ be nuclear and α a stable exponent sequence. Then $\Lambda_\infty(\alpha)' = \Delta(\Lambda_\infty(\alpha)) \subset \Delta(E) \Leftrightarrow E$ is isomorphic to a subspace of $\Lambda_\infty(\alpha)^N$.*

We can consider this theorem as a characterization of subspaces of $\Lambda_\infty(\alpha)^N$. Certainly $\Lambda_\infty(\alpha)$ is a complemented subspace of $\Lambda_\infty(\alpha)^N$, but whereas $\Lambda_\infty(\alpha)^N$ has no continuous norm, the topology of $\Lambda_\infty(\alpha)$ is defined by an increasing sequence of norms. So to characterize subspaces or quotient spaces of $\Lambda_\infty(\alpha)$ or of $\Lambda_1(\alpha)$, we need tools other than diametral dimension.

To simplify the argument in what follows let us assume $\alpha_m < \alpha_{m+1} \ \forall\, m \in N$. Let

$$x = (x_n) \in U_{k+1} \Leftrightarrow \sum^{\infty} |x_n| e^{(k+1)\alpha_n} \leq 1$$

For each m, let $y^m = (x_1, \dots, x_m, 0, 0, \dots)$ and $z^m = x - y^m = (0, \dots, 0, x_{m+1}, \dots)$. Then

$$\|z^m\|_k = \sum_{i=m+1} |x_i| e^{k\alpha_i} = \sum_{i=m+1}^{\infty} |x_i| \frac{e^{(k+1)\alpha_i}}{e^{\alpha_i}} \le e^{-(\alpha_{m+1})}$$

and for $j > k$

$$\begin{aligned} \|y^m\|_j &= \sum^{m} |x_i| e^{j\alpha_i} = \sum^{m} |x_i| e^{(k+1)\alpha_i} e^{(j-k-1)\alpha_i} \\ &\le e^{(j-k-1)\alpha_m}. \end{aligned}$$

So we simply get for any m and $j > k$

$$U_{k+1} \subset e^{(j-k-1)\alpha_m} U_j + \frac{1}{e^{\alpha_{m+1}}} U_k.$$

For any $r \ge e^{\alpha_1}$, let m be the smallest integer such that $r \le e^{\alpha_{m+1}}$. Here we are using the fact $\lim \alpha_n = \infty$. Then we have the following

$$U_{k+1} \subset C r^{j-k-1} U_j + \frac{1}{r} U_k$$

for some constant $C > 0$ and for all $r > 0$. The assumption $\alpha_m < \alpha_{m+1}$ was really not necessary, although it made our calculations neater.

Let $U_1 \supset U_2 \supset \cdots \supset U_k$ be a base of neighborhoods of E. We say E satisfies the *condition* (Ω) if $\forall k \;\; \exists p \; \forall j \;\; \exists \mu \;\; \exists C > 0$ with

$$U_p \subset C r^{\mu} U_j + \frac{1}{r} U_k, \quad \forall r > 0.$$

We have proved that $\Lambda_\infty(\alpha)$ satisfies the condition (Ω). Note that if E has (Ω) and F is a quotient space of E, then F has (Ω) also. This follows from the fact that $(Q(U_k))$ is a base of neighborhoods of F if $Q : E \to F$ is a quotient map.

Somewhat dual to (Ω), we have the so-called *dominating norm condition.* Again E is an F-space and $(\| \; \|_k)$ a sequence of seminorms defining its topology. We say E has *property (DN)* if $\exists k_0 \;\; \forall k \;\; \exists p$ and $C > 0$ with $\|x\|_k^2 \le C \|x\|_{k_0} \;\; \|x\|_p, \forall \;\; x \in E$. It is easy to see that $\| \cdot \|_{k_0}$ is in fact a norm and without loss of generality we can assume the topology of E is defined by an increasing sequence of norms, if E has (DN). Further (DN) is inherited by subspaces. From $\|e_n\|_k = e^{k\alpha_n}$, for (e_n) in $\Lambda_\infty(\alpha)$, we can easily get that $\Lambda_\infty(\alpha)$ has property (DN).

Conditions (DN) and (Ω) were introduced by Vogt [41] and Vogt and Wagner [46], [47] to characterize subspaces and quotient spaces of nuclear, stable power series spaces. However these and similar conditions are also essential in Vogt's ground breaking work in lifting or extension theorems in the category of F-spaces or more generally in examining when the functor Ext vanishes (cf. [44], [45], [47]), see also [30]. (see also [30] We will go into this rich theory so far it is necessary to describe the characterizations of subspaces and quotient spaces of nuclear, stable power series spaces of infinite type.

Given a short exact sequence of nuclear F-spaces

$$0 \longrightarrow E \xrightarrow{i} G \xrightarrow{q} F \longrightarrow 0.$$

This means $i : E \to G, \quad q : G \to F$ are continuous linear maps, $q^{-1}(0) = i(E)$, $i^{-1}(0) = 0$ and $q(G) = F$. Hence $i : E \to i(E)$ is an isomorphism by the open-mapping theorem, since $i(E) = q^{-1}(0)$ is a closed subspaces and F is isomorphic to a quotient space of G. We have the following lifting theorem of Vogt.

Theorem 5.2. *Let* $0 \to E \to G \xrightarrow{q} F \to 0$ *be an exact sequence of nuclear Fréchet space. Assume* H *is a nuclear Fréchet space which has* (DN) *and* E *has* (Ω). *Then every continuous linear map* $t : H \to G$ *can be lifted to a map* $\tilde{t} : H \to E$*; that is* $q\tilde{t} = t$.

Let α be a stable exponent sequence and $\Lambda(\alpha)$ a nuclear power series space, which can be finite or infinite type. Then we have

Proposition 5.3. *There is an exact sequence*

$$0 \to \Lambda(\alpha) \xrightarrow{i} \Lambda(\alpha) \xrightarrow{q} \Lambda(\alpha)^N \to 0$$

The construction of this exact sequence is not so difficult in case $\Lambda(\alpha) = s$ [47], but to prove it in this generality quite a lot of elaborate calculations are needed [47]. We first note that this rather technical looking result says that $\Lambda(\alpha)$ and $\Lambda(\alpha)^N$ have same quotient spaces. Further stability of α is necessary, since the result gives $\Delta(\Lambda(\alpha)) = \Delta(\Lambda(\alpha)^N)$ which can be true only if α is stable.

We can now prove in an elegant manner the following theorem (cf. [46], [47]), characterizing subspaces of power series spaces of infinite type.

Theorem 5.4. *Let* $\Lambda_\infty(\alpha)$ *be nuclear and stable.* E *is isomorphic to a subspace of* $\Lambda_\infty(\alpha)$ *if and only if* $\Lambda_\infty(\alpha)' \subset \Delta(E)$ *and* E *has* (DN).

Proof. By 5.3. Proposition we have an exact sequence

$$0 \to \Lambda_\infty(\alpha) \to \Lambda_\infty(\alpha) \xrightarrow{q} \Lambda_\infty(\alpha)^N \to 0.$$

By 5.1 Theorem we have an imbedding $t : E \to \Lambda_\infty(\alpha)^N$, i.e. t is continuous, 1-1 and $t(E)$ is closed. Since E has (DN) and $\Lambda_\infty(\alpha)$ has (Ω) there is a continuous linear map $\tilde{t} : E \to \Lambda_\infty(\alpha)$ such that $q\tilde{t} = t$ by 5.2. Theorem. Clearly $\tilde{t}$ is 1-1. Let $\lim \tilde{t}(x_n) = y \in \Lambda_\infty(\alpha)$. Then $\lim q\tilde{t}(x_n) = \lim t(x_n) = q(y)$. Since $t : E \to t(E)$ is an isomorphism, we know $\lim t(x_n)$ exists $\Leftrightarrow \lim x_n$ exists. Let $x = \lim x_n$. Then $\lim \tilde{t}(x_n) = \tilde{t}(x) = y$. So $\tilde{t}$ has closed range and thus by the open mapping theorem, $\tilde{t}$ is an imbedding. □

As an immediate corollary we get that a nuclear Fréchet space which has property (DN), is isomorphic to a subspace of the space s of rapidly decreasing sequences.

Suppose we have the following diagram of F-spaces and continuous linear maps

$$\begin{tikzcd}
 & & & 0 & \\
0 \arrow[r] & E_1 \arrow[r, "i_1"] & E_2 \arrow[r, "q_1"] & Q \arrow[r] \arrow[u] & 0 \\
 & & & F_2 \arrow[u, "q_2"'] \arrow[ul, "t"] & \\
 & & & F_1 \arrow[u, "i_2"'] & \\
 & & & 0 \arrow[u] &
\end{tikzcd}$$

such that the row and the column are both exact and the diagram is commutative, that is $q_1 t = q_2$. We will now do some diagramm chasing as in homological algebra.

We first define $i : F_1 \to E_1 \oplus F_2$ as follows.

Given $z \in F_1$. $q_2 i_2(z) = 0 = q_1 t i_2(z)$. Since $q_1^{-1}(0) = i_1(E_1)$, there is a unique $x \in E_1$ with $i_1(x) = t i_2(z)$. Define now $f : F_1 \to E_1$ by setting $f(z) = x$. Then $i_1(f(z)) = t i_2(z)$. Now let us define $i : F_1 \to E_1 \oplus F_2$ by $i(z) = (f(z), -i_2(z))$. i is certainly 1-1. Next, define $q : E_1 \oplus F_2 \to E_2$ by setting $q(x, y) = i_1(x) + t(y)$. $qi(z) = i_1(f(z) - t i_2(z) = 0$. So range of i is contained in the kernel of q. If $q(x, y) = 0$ then $i_1(x) = -t(y)$ so $q_1 i_1(x) = 0 - q_1 t(y) = -q_2(y_2)$. So $y = i_2(z)$ some $z \in F_1$. Hence $t(y) = t i_2(z) = i_1(f(z))$ and since i_1 is 1-1, $x = f(z)$ and $q^{-1}(0)$ is contained in the range of i.

Further, if $w \in E_2$, then $q_1(w) = q_2(y)$ for some $y \in F_2$. Since $q_1(ty) = q_2(y) = q_1(w)$. So $t(y) - w \in i_1(E_1)$ and therefore there is $x \in E_1$ $t(y) + i_1(x) = w = q(x, y)$. Therefore q is a surjection. Thus we have constructed by this procedure the following exact sequence

$$0 \to F_1 \to E_1 \oplus F_2 \to E_2 \to 0.$$

Lemma 5.5. *If $\Lambda_\infty(\alpha)' \subset \Delta(E)$, then there is a closed subspace $\tilde{E}$ of $\Lambda_\infty(\alpha)$ and an exact sequence*

$$0 \to \Lambda_\infty(\alpha) \to \tilde{E} \to E \to 0$$

Proof. We go back to the exact sequence

$$0 \to \Lambda_\infty(\alpha) \to \Lambda_\infty(\alpha) \xrightarrow{q} \Lambda_\infty(\alpha)^N \to 0.$$

We have an imbedding $t : E \to \Lambda_\infty(\alpha)^N$ by 5.3. Prop. Let $\tilde{E} = q^{-1}(t(E))$. Identifying E with $t(E)$ we have the result. ☐

We are now ready to characterize quotient spaces of $\Lambda_\infty(\alpha)$. In fact our result yields more.

Theorem 5.6. *(*[47]*, 3.3. Lemma). Let $\Lambda_\infty(\alpha)' \subset \Delta(E)$ where $\Lambda_\infty(\alpha)$ is a stable nuclear space. If E has property (Ω), then there is a subspace $\tilde{E}$ of $\Lambda_\infty(\alpha)$ and an exact sequence*

$$0 \to \Lambda_\infty(\alpha) \to \Lambda_\infty(\alpha) \to E \oplus \tilde{E} \to 0.$$

In particular, E is isomorphic to a quotient space of $\Lambda_\infty(\alpha)$.

Proof. We imbed $i : E \to \Lambda_\infty(\alpha)^N$ by 5.1 Thm and let $Q = \Lambda_\infty(\alpha)^N / i(E)$. We apply our lemma to Q to get the column in the following commutative diagramm. Note that we have also used 5.2. Thm.

$$\begin{tikzcd}
 & & & 0 & \\
0 \arrow[r] & E \arrow[r] & \Lambda_\infty(\alpha)^N \arrow[r] & Q \arrow[r] \arrow[u] & 0 \\
 & & & \tilde{E} \arrow[u] \arrow[ul] & \\
 & & & \Lambda_\infty(\alpha) \arrow[u] & \\
 & & & 0 \arrow[u] &
\end{tikzcd}$$

We apply our homological algebra procedure to get the line in the following commutative diagram

$$\begin{tikzcd}
 & & & 0 & \\
0 \arrow[r] & \Lambda_\infty(\alpha) \arrow[r] & E \oplus \tilde{E} \arrow[r] & \Lambda_\infty(\alpha)^N \arrow[r] \arrow[u] & 0 \\
 & & & \Lambda_\infty(\alpha) \arrow[u] \arrow[ul] & \\
 & & & \Lambda_\infty(\alpha) \arrow[u] & \\
 & & & 0 \arrow[u] &
\end{tikzcd}$$

Now once more we apply our homological algebra procedure to arrive at the conclusion. ☐

Let us continue the same line of argument further. Suppose E is isomorphic to a subspace and a quotient space of $\Lambda_\infty(\alpha)$. We have then a subspace $\tilde{E}$ of $\Lambda_\infty(\alpha)$ and an exact sequence

$$0 \to \Lambda_\infty(\alpha) \to \Lambda_\infty(\alpha) \to E \oplus \tilde{E} \to 0.$$

Since $E \oplus \tilde{E}$ has (DN), this sequence splits. That is $E \oplus \tilde{E}$ and therefore E is a complemented subspace of $\Lambda_\infty(\alpha)$. We have proved now the following result.

Theorem 5.7. *Let $\Lambda_\infty(\alpha)' \subset \Delta(E)$ where $\Lambda_\infty(\alpha)$ is a stable nuclear space. If E has properties (DN) and (Ω), then it is isomorphic to a complemented subspace of $\Lambda_\infty(\alpha)$.*

So far we have dealt with the problem of characterizing subspaces and quotient spaces of nuclear, stable power series spaces of infinite type. We will now consider finite type power series spaces and characterize again subspaces and quotient spaces. The results in this case are similar to the case of power series spaces of infinite type, however the methods used in proving the theorems characterizing subspaces or quotient spaces are somewhat different. (cf. [46])

We say E has *property* $(\underline{DN})$ if

$$\exists k_0 \;\; \forall k \;\; \exists \;\; 0 < \lambda < 1 \;\; \exists p, \exists C > 0$$

with

$$\|x\|_k \leq C \|x\|_{k_0}^{\lambda} \; \|x\|_p^{1-\lambda}, \quad x \in E.$$

Remarks.

(1) It is easy to see that $\| \; \|_{k_0}$ is indeed a norm and so we can assume that the topology of E is defined by an increasing sequence of norms. Also (DN) implies $(\underline{DN})$; hence we call $(\underline{DN})$ the *weak dominating norm property.*

(2) $\Lambda_1(\alpha)$ satisfies $(\underline{DN})$. Again $(\underline{DN})$ is inherited by subspaces. So a subspace of $\Lambda_1(\alpha)$ also has $(\underline{DN})$.

The following is the counterpart of Thm. 5.4., which is also due to Vogt [46].

Theorem 5.8. *Let $\Lambda_1(\alpha)$ be stable and nuclear. Then E is isomorphic to a subspace of $\Lambda_1(\alpha)$ if and only if $\Lambda_1(\alpha) \subset \Delta(E)$ and E has $(\underline{DN})$.*

E has property $(\overline{\Omega})$ if $\forall p\ \exists q\ \forall k\ \exists C > 0$ such that

$$U_q \subset CrU_k + \frac{1}{r}U_p, \quad r > 0.$$

Remarks.

(1) We have $(\overline{\Omega})$ implies (Ω) and $(\overline{\Omega})$ is inherited by quotient spaces.
(2) We can also show that $\Lambda_1(\alpha)$ has property $(\overline{\Omega})$ and therefore every quotient space of $\Lambda_1(\alpha)$ also has $(\overline{\Omega})$.
(3) Characterization of stable finite power series spaces in terms of these invariants were given in [42] (see cf. [1]).

Theorem 5.9. *Let $\Lambda_1(\alpha)$ be stable and nuclear. Then E is isomorphic to a quotient space of $\Lambda_1(\alpha)$ if and only if $\Lambda_1(\alpha) \subset \Delta(E)$ and E has $(\overline{\Omega})$.*

We have already noted that every continuous linear map $T : \Lambda_1(\alpha) \to \Lambda_\infty(\beta)$ is compact. Hence no infinite dimensional quotient space of $\Lambda_1(\alpha)$ can be isomorphic to a subspace of $\Lambda_\infty(\beta)$. In particular, although $\Lambda_\infty(\beta)$ has (Ω), it does not satisfy $(\overline{\Omega})$. For subspaces things are quite different. Since (DN) implies $(\underline{DN})$, we have from Theorem 5.8. that $\Lambda_\infty(\beta)$ is isomorphic to a subspace of $\Lambda_1(\alpha)$ if $\Lambda_1(\alpha) \subset \Lambda_\infty(\beta)'$, provided $\Lambda_1(\alpha)$ is nuclear and stable. In particular $\Lambda_\infty(\alpha)$ is isomorphic to a subspace of $\Lambda_\infty(\alpha)$. However we really do not need stability in this context as shown by Nurlu [29].

6. COMPLEMENTED G_∞-SPACES

In general we shall deal with the following problem: Given a nuclear Fréchet space E and assume $\Delta(E) \subset \Lambda_\infty(\alpha)'$, where α is stable. When can we say that $\Lambda_\infty(\alpha)$ is isomorphic to a complemented subspace of E? The results given in this section are due to Aytuna, Krone and Terzioğlu, starting with [3] and continuing in [4] [5].

Let $\lambda(A)$ be a nuclear G_∞-space and E a locally convex space. We call a linear, continuous map $i : \lambda(A) \to E$ a *local imbedding* if there is a continuous seminorm $\|\cdot\|$ on E and a sequence $\sigma = (\sigma_n)$ which satisfies the following condition: $\sigma_n > 0\ \ \forall n,\ \ \sigma_n = 0(a_n),\ \ 1/\sigma_n = 0(b_n)$ for some $a, b \in A$, such that

$$|x|_\sigma = \sum_{n=1}^{\infty} |\xi_n|\sigma_n \le \|i(x)\|, \quad x = (\xi_n) \in \lambda(A).$$

Remarks.

(1) Typically the sequence $e = (1, 1, \ldots)$ satisfies the condition required from σ in the definition. In fact this is how local imbedding was defined originally in [3].

(2) A local imbedding is certainly 1-1 and so $\|\cdot\|$ is in fact a norm on the range of i.

(3) An imbedding of $\lambda(A)$ into E is certainly a local imbedding, but the converse is false as the following example shows. Take $\sigma = e$ and let $i : \Lambda_\infty(\alpha) \to \Lambda_1(\alpha)$ be defined by $i(\xi_n) = (2^{\alpha_n}\xi_n)$. i is a local imbedding but it is a compact map.

(4) If $i : \lambda(A) \to E$ is a local imbedding and $j : E \to F$ an imbedding, then $ji : \lambda(A) \to F$ is a local imbedding.

Let $i : \lambda(A) \to E$ be a local imbedding where $|x|_\sigma \leq \|i(x)\|$. Given $a \in A$, let $c_n = a_n/\sigma_n$. From $\sigma_n c_n = a_n$ we have $c_n = 0(d_n)$ for some $d = (d_n) \in A$. Define now $D_c : \lambda(A) \to \lambda(A)$ by $D_c(x) = (x_n c_n)$. Then D_c is continuous and $|D_c(x)|_\sigma = |x|_a$ for every $x \in \lambda(A)$. Then $i_a = iD_c$ and

$$|x|_a = |D_c x|_\sigma \leq \|i_a(x)\|.$$

Hence if there is a local imbedding $i : \lambda(A) \to E$, then there is a continuous seminorm $\|\cdot\|$ on E such that for every $a \in A$ we have a local imbedding $i_a : \lambda(A) \to E$ which satisfies $|x|_a \leq \|i_a(x)\|, \quad x \in \lambda(A)$. Then, if we define $T(x) = (i_a(x))$, we have an imbedding of $\lambda(A)$ into E^A.

Proposition 6.1. *If there is a local imbedding of $\lambda(A)$ into E, then $\lambda(A)$ is isomorphic to a subspace of E^A. If $\lambda(A)$ is in particular a Fréchet space, then it is isomorphic to a subspace of E^N.*

We now state our generic theorem and at least indicate its proof.

Theorem 6.2. *Let $\lambda(A)$ be a nuclear, stable G_∞-space and assume there is a local imbedding of $\lambda(A)$ into a locally convex space E. Then each one of the following conditions implies that $\lambda(A)$ is isomorphic to a complemented subspace of E.*

(1) *$\lambda(A)$ is Fréchet and E is isomorphic to a closed subspace of $\lambda(A)$.*

(2) *E is isomorphic to a closed subspace of $\lambda(A)$*

(3) *E is isomorphic to a sequentially complete quotient space of $\lambda(A)$.*

Let $i : \lambda(A) \to E$ be a local imbedding with

$$|x|_c = \sum |\xi_n| \leq |||i(x)|||, \quad x = (\xi_n) \in \lambda(A)$$

where $|||\cdot|||$ is given by a semi-inner product $(\cdot|\cdot)$. We can assume this since each one of our assumptions implies that E is nuclear. Let E_n be the subspace of E spanned by $i(e_1), \ldots, i(e_{2n})$. We note that E_n has dimension $2n$. We let (f_v) be a sequence in E which will be specified later and by the usual Gram-Schmidt process choose $g_n \in E_n$ such that

$$\begin{aligned} (g_n|g_j) &= 0, \qquad j = 1, \ldots, n-1 \\ |||g_n||| &= 1 \\ (g_n|f_v) &= 0, \qquad v = 1, \ldots, n. \end{aligned}$$

Let

$$g_n = i\left(\sum_{j=1}^{2n} \mu_j^n e_j\right)$$

and so

$$\sum^{2n} |\mu_j| \leq |||g_n||| = 1.$$

Further for each continuous seminorm $\| \cdot \|$ on E, we can find $c \in A$ with

$$\begin{aligned} \|g_n\| &\leq \rho \sum^{2n} |\mu_j| c_j \leq \rho \left(\sum^{2n} |\mu_j^n|\right) c_{2n} \\ &\leq \tilde{\rho} a_n \end{aligned}$$

for some $\tilde{\rho} > 0$, where $a \in \lambda(A)$, satisfying $c_{2n} = 0(a_n)$. So if we define a new map $j : \lambda(A) \to E$ by $j(e_n) = g_n$, we see that j is continuous.

Now we have to select (f_v) in each of the three cases. Let us only indicate how this in done in case $E \subset \lambda(A)$. In this case we simply let $f_v = e_v$. So if $x = (\xi_j) \in E$, then there is some $\tilde{a} \in A$ with $|||f_v||| = 0(\tilde{a}_v)$ and

$$|(g_n|x)|a_n \leq \rho \sum_{v>n} |\xi_v| \tilde{a}_v a_n \leq \rho \sum_{v>n} |\xi_j| \tilde{a}_v a_v.$$

Using nuclearity we choose $b \in A$ with

$$a_v \tilde{a}_v = 0\left(\frac{1}{v^2} b_v\right)$$

to get

$$|(g_n|x)|a_n \leq \frac{\tilde{\rho}}{n^2} \|x\|_b.$$

So if we define now $P : E \to E$ by

$$P(x) = \sum (g_n|x) g_n$$

we see $P(g_n) = g_n$, P is a continuous projection with $P(E)$ equal to the closure of the span of $\{g_n : n \in N\}$. It remains to prove that $j : \lambda(A) \to E$ is an isomorphism onto $P(E)$.

To get some important but rather immediate consequences of our generic theorem, we specialize to power series spaces of infinite type. Throughout this section from now on, we assume $\Lambda_\infty(\alpha)$ is *stable and nuclear*. Our first result is a direct consequence of Theorem 5.1. and Theorem 6.2.

Corollary 6.3. *Let E be a Fréchet space with $\Lambda_\infty(\alpha)' \subset \Delta(E)$. If there is a local imbedding of $\Lambda_\infty(\alpha)$ into E, then E has a complemented subspace isomorphic to $\Lambda_\infty(\alpha)$. In particular, if there is a local imbedding of s into a nuclear Fréchet space E, E has a complemented subspace isomorphic to s.*

Let E be a nuclear Fréchet space which has properties (DN) and (Ω). We know that the diametral dimension of E is equal to the diametral dimension of some power series space of infinite type. Assume $\Delta(E \oplus E) = \Delta(E)$. Then $\Delta(E) = \Delta(\Lambda_\infty(\alpha)) = \Lambda_\infty(\alpha)'$ and α is stable. So by Theorem 5.7. E is isomorphic to a complemented subspace of $\Lambda_\infty(\alpha)$. We have also the following lemma ([5]). (cf. [39]).

Proposition 6.4. *Let E be a nuclear space with (DN) and (Ω) and assume $\Delta(E) \subset \Delta(\lambda(B))$, where $\lambda(B)$ is a G_∞-Köthe space. Then there is a local imbedding of $\lambda(B)$ into E.*

Going back to the previous set-up, we now know that there is a local imbedding of $\Lambda_\infty(\alpha)$ into E. Hence by Corollory 6.3., E has a complemented subspace which is isomorphic to $\Lambda_\infty(\alpha)$. Applying Vogt's decomposition principle, we get the following result.

Corollary 6.5. *Let E be isomorphic to a complemented subspace of s. If $\Delta(E \times E) = \Delta(E)$, then E is isomorphic to some $\Lambda_\infty(\alpha)$. In particular E has a basis.*

7. Duality and Smooth Sequence Spaces

Let $\lambda(Q)$ be a nuclear G_1-space and

$$A_Q = \{x \in \lambda(Q) : \ 1 \leq x_1 \leq x_2 \leq \cdots\}.$$

Then $\lambda(A_Q)$ is a nuclear G_∞-space and the strong dual $\lambda(Q)'_b$ is topologically isomorphic to a subspace of $\lambda(Q)$. We have the following result.

Proposition 7.1. *The dual $\lambda(Q)'_b$ of a nuclear G_1-space $\lambda(Q)$ is isomorphic to a dense subspace of the nuclear G_∞-space $\lambda(A_Q)$. If $\lambda(Q)$ is stable then $\lambda(A_Q)$ is also stable. $\lambda(Q)$ is barrelled if and only if $\lambda(Q)'_b = \lambda(A_Q)$. In particular, the dual of a nuclear power series space of finite type is a nuclear G_∞-space.*

For the duals of G_∞-spaces we have a result of the same nature. However there is a minor point which requires some care. If we let A be the set of all positive non-decreasing sequences, then the G_∞-space $\lambda(A)$ is equal to φ, the space of all sequences with only finitely many non-zero terms. Its dual is ω which cannot be represented as a G_1 space. So we assume now $\lambda(A) \neq \varphi$ and let

$$Q_A = \{x \in \lambda(A) : \ 0 < x_{n+1} \leq x_n\}.$$

Then $\lambda(Q_A)$ is a nuclear G_1-space.

Proposition 7.2. *Let $\lambda(A)$ be a nuclear, G_∞-space and $\varphi \neq \lambda(A)$. Then $\lambda(A)'_b$ is isomorphic to a dense subspace of the nuclear G_1-space $\lambda(Q_A)$. If $\lambda(A)$ is stable, then $\lambda(Q_A)$ is also stable. $\lambda(A)$ is barrelled if and only if $\lambda(A)'_b = \lambda(Q_A)$. In particular the dual of a nuclear power series space of infinite type is a nuclear G_1-space.*

In this duality set-up, we seek the concept which is dual to the concept of local imbeddings. Let $\lambda(Q)$ be a nuclear G_1-space, for $x = (x_j) \in \lambda(Q), \ x_j \geq 0$ let

$$B_x = \{y : |y_j| \leq x_j \quad \forall j \in N\}.$$

It is easy to see that each such set is bounded. In fact one can obtain a base of bounded subsets of $\lambda(Q)$ in this manner. Assume now $x_j > 0 \ \forall j$ and x and $(1/x_j) \in \lambda(Q)$. A continuous linear map $h : E \to \lambda(Q)$ is called a local quotient if there is some $B \in \mathcal{B}(E)$ with $B_x \subset \overline{h(B)}$.

Proposition 7.3. *Let* $\lambda(Q)$ *be a nuclear barrelled* G_1*-space and* $h : E \to \lambda(Q)$ *a local quotient. Then its transpose* $h' : \lambda(Q)'_b \to E'_b$ *is a local imbedding.*

After these preparations, we can now apply our generic result, Theorem 6.2., to obtain the following theorem ([39]).

Theorem 7.4. *Let* $\Lambda_1(\alpha)$ *be a stable and nuclear. Let* E *be an* F*-space such that there is a local quotient from* E *into* $\Lambda_1(\alpha)$*. If* E *is either isomorphic to a subspace of* $\Lambda_1(\alpha)$ *or to a quotient space of* $\Lambda_1(\alpha)$*, then* E *has a complemented subspace isomorphic to* $\Lambda_1(\alpha)$ *itself.*

8. Associated Exponent Sequence

In this section we will summarize the highlights of a joint work by Aytuna, Krone and Terzioğlu [5]. Let E be a nuclear Fréchet space with properties $(\underline{DN})$ and (Ω). Then we can find an exponent sequence $\epsilon = (\epsilon_n)$ such that

$$\Lambda_1(\epsilon) \subset \Delta(E) \subset \Lambda_\infty(\epsilon)'.$$

Further

$$\begin{aligned} \Lambda_1(\alpha) \subset \Delta(E) \quad &\Leftarrow \quad \Lambda_1(\alpha) \subset \Lambda_1(\epsilon) \\ \Delta(E) \subset \Lambda_\infty(\alpha)' \quad &\Leftarrow \quad \Lambda_\infty(\epsilon)' \subset \Lambda_\infty(\alpha)'. \end{aligned}$$

We call ϵ the *exponent sequence associated* to E. From what we have stated above, we can see that it is unique up to equivalence.

Lemma 8.1. *Let* E *be a nuclear Fréchet space with* $(\underline{DN})$ *and* (Ω)*. If* $\lambda(A)$ *is a* G_∞*-Köthe space with* $\Delta(E) \subset \lambda(A)'$*, then there is a local imbedding of* $\lambda(A)$ *into* E*. In particular, there is a local imbedding of* $\Lambda_\infty(\epsilon)$ *into* E*.*

Let E be again a nuclear Fréchet space with $(\underline{DN})$ and (Ω). Suppose now $\Delta(E) = \Lambda_\infty(\epsilon)'$ and ϵ is stable. We then know that E is isomorphic to a quotient space of $\Lambda_\infty(\epsilon)$ (cf. Theorem 5.6.) and by our Lemma there is a local imbedding of $\Lambda_\infty(\epsilon)$ into E. As a corollary of our generic Theorem 6.2. we get the following result.

Theorem 8.2. *Let* E *be a nuclear Fréchet space with* $(\underline{DN})$ *and* (Ω)*. Assume* $\Delta(E) = \Lambda_\infty(\epsilon)'$*. If* ϵ *is stable, then* E *has a complemented subspace isomorphic to* $\Lambda_\infty(\epsilon)$*.*

Remarks. In the context of our theorem, the spaces E and $\Lambda_\infty(\epsilon)$ have the same quotient spaces.

We have now an imbedding theorem

Theorem 8.3. *Let E be a nuclear Fréchet space with $(\underline{DN})$ and (Ω) and let ϵ be the associated exponent sequence. Assume $\Lambda_1(\epsilon)$ is nuclear. If Y is isomorphic to a subspace of $\Lambda_1(\epsilon)$ and Y has (DN), then Y is isomorphic to a subspace of E. If ϵ is stable, then $\Lambda_\infty(\epsilon)$ itself is isomorphic to a subspace of E.*

Let us now apply our results first to spaces of analytic functions. For a Stein manifold M of dimension d, let $\mathcal{O}(M)$ be the space of analytic functions on M with the topology of uniform convergence on compact subsets of M. $\mathcal{O}(M)$ is a nuclear Fréchet space. Let

$$\Delta^d = \{(z_j) \in \mathbb{C}^d : |z_j| < 1, \quad j = 1, 2, \dots, d\}.$$

In particular, $\mathcal{O}(\Delta^d)$ is isomorphic to the power series space $\Lambda_1(n^{1/d})$ and the space of entire functions $\mathcal{O}(\mathbb{C}^d)$ is isomorphic to $\Lambda_\infty(n^{1/d})$. ([34]) Since $\mathcal{O}(M)$ is isomorphic to a subspace of $\mathcal{O}(\Delta^d)$, it has property $(\underline{DN})$. ([6]) By the Oka-Cartan theorem, $\mathcal{O}(M)$ is isomorphic to a quotient space of some $\mathcal{O}(\mathbb{C}^m)$ and so it has also (Ω). In fact the space $\mathcal{O}(M)$ has $(n^{1/d})$ as its associated exponent sequence. Since $(n^{1/d})$ is stable, we have the following consequence of Theorem 8.3.

Theorem 8.4. *$\mathcal{O}(\mathbb{C}^d)$ is isomorphic to a subspace of $\mathcal{O}(M)$.*

We can apply our previous results to the space of analytic functions and obtain the following theorems

Theorem 8.5. *If $\Delta(\mathcal{O}(M)) = \Delta(\mathcal{O}(\mathbb{C}^d))$ then the following are true:*

a) *$\mathcal{O}(M)$ is isomorphic to a subspace of $\mathcal{O}(\mathbb{C}^d)^N$.*
b) *$\mathcal{O}(M)$ is isomorphic to a quotient space of $\mathcal{O}(\mathbb{C}^d)$.*
c) *$\mathcal{O}(M)$ is isomorphic to $\mathcal{O}(\mathbb{C}^d) \oplus F$, where F is isomorphic to a quotient space of $\mathcal{O}(\mathbb{C}^d)$.*

Let us examine the case $\mu = \Delta^r \times \mathbb{C}^{k-r}$ where $1 \leq r < k$. By a theorem proved independently by Djakov and Zahariuta (cf. [11]), we know that $O(\Delta^r \times \mathbb{C}^{k-r})$ is isomorphic to $\mathcal{O}(\Delta \times \mathbb{C}^{k-1})$. $\Delta \times \mathbb{C}^{k-1}$ is a k-dimensional complete Reinhardt domain and by a result in [7] (Theorem 1.5.) its diametral dimension is equal to the diametral dimension of $\mathcal{O}(\mathbb{C}^k)$. In fact, $\mathcal{O}(\Delta \times \mathbb{C}^{k-1})$ is isomorphic to the complete tensor product $\mathcal{O}(\Delta \times C)\widetilde{\otimes}\mathcal{O}(\mathbb{C}^{k-2})$.

Hence we can fix our attention on $\mathcal{O}(\Delta \times \mathbb{C})$. By Theorem 8.5. there is a certain quotient space X of $\mathcal{O}(\mathbb{C}^2)$ such that $\mathcal{O}(\Delta \times \mathbb{C})$ is isomorphic to $\mathcal{O}(\mathbb{C}^2) \oplus X$. When we examine the nature of this space X, we can see that it does not have (DN), but we have the isomorphism

$$\mathcal{O}(\Delta^r \times \mathbb{C}^{k-r}) \simeq \mathcal{O}(\mathbb{C}^k) \oplus (X\widetilde{\otimes}\mathcal{O}(\mathbb{C}^{k-2})) \quad \text{for} \quad 1 \leq r < k.$$

However in the one dimensional case, we have $\mathcal{O}(G) \simeq \mathcal{O}(\mathbb{C})$ if and only if $\Delta(\mathcal{O}(G)) = \Delta(\mathcal{O}(\mathbb{C}))$, where G is a domain in $\mathbb{C}$ ([5]; Cor. 1.7.)

Going back to Stein manifolds, we know that if $\mathcal{O}(M)$ has property (DN), then its diametral dimension is equal to $\Lambda_\infty(n^{1/d})'$. ([38]) So we have the following result $(i) \Leftrightarrow (iii)$ was proved independently by several authors including Zaharyuta, Aytuna, Vogt. (cf. also [2], [6]).

Theorem 8.6. *For a d-dimensional Stein manifold M, the following conditions are equivalent*

(i) *$\mathcal{O}(M)$ has (DN).*
(ii) *$\mathcal{O}(M)$ is isomorphic to $\mathcal{O}(\mathbb{C}^d)$.*
(iii) *Every bounded plurisubharmonic function on M is constant.*

Let $P(D)$ be an elliptic linear partial differential operator with constant coefficients on $R^k,\ k \geq 2$. Let $V \subset R^k$ be open and connected and let

$$N_p(V) = \{f \in C^\infty(V) : P(D)f = 0\}.$$

The situation in this case resembles the spaces of analytic functions. First of all we have the isomorphisms

$$N_p(B) \simeq \Lambda_1(n^{\frac{1}{k-1}}), \quad N_p(R^k) \simeq \Lambda_\infty(n^{\frac{1}{k-1}})$$

where B is any open, convex and bounded subset of R^k. $N_p(V)$ has properties $(\underline{DN})$ and (Ω) and its associated exponent sequence is $(n^{\frac{1}{k-1}})$. [cf. [48], [23]] So we have the following result:

Theorem 8.7. *Let V be an open connected subset of R^k. Then $N_p(V)$ is isomorphic to a subspace of $N_p(B)$, where B is open, convex and bounded. $N_p(R^k)$ is isomorphic to a subspace of $N_p(V)$. $N_p(V)$ has property (DN) if and only if it is isomorphic to $N_p(R^k)$.*

Let E be a nuclear Fréchet space and $H(E'_b)$ be the space of holomorphic functions on the dual space E'_b, equipped with the topology of uniform convergence on compact subsets of E'_b. $H(E'_b)$ is a nuclear Fréchet space. See for example [24] Börgens, Meise and Vogt ([9], [10]) have proved that $H(\Lambda_\infty(\alpha)'_b)$ is isomorphic to a power series space $\Lambda_\infty(\beta(\alpha))$. This exponent sequence $\beta(\alpha)$ depends on α and *it is always stable.* Further $\beta(\log(n+1))$ is equivalent to $(\log(n+1))$ and so $H(s'_b)$ is isomorphic to s itself [9]. Further we can prove ([39]; 5.5. Prop.) that if there is a local imbedding of $\Lambda_\infty(\alpha)$ into E, then there is a local imbedding of $\Lambda_\infty(\beta(\alpha))$ into $H(E'_b)$. With this fact at our disposal we can prove that following surprising result.

Theorem 8.8. *Let E be a complemented subspace of s. Then $H(E'_b)$ is isomorphic to $\Lambda_\infty(\beta(\alpha))$ where $\Delta(E) = \Lambda_\infty(\alpha)'$.*

Note that in the theorem, we do not know whether E has a basis, but we get that the much larger space $H(E'_b)$ always has a basis. E is certainly a complemented subspace of $H(E'_b)$, namely the functions which are linear. So we have the following problem.

Question. Let E be a complemented subspace of s. Can one find a tame projection on $H(E'_b)$ whose range is isomorphic to E?

A positive answer to this question would mean a solution to the Problem 4.5. If E has $(\underline{DN}), (\Omega)$ and α is its associated exponent sequence. Then it can be shown that $\beta(\alpha)$ is the associated exponent sequence of $H(E'_b)$. This fact can yield more applications of our results in the context of infinite-dimensional holomorphy (see [39]). However the spaces $H(V), V$ an open subset of E'_b, need to be researched further.

To conclude this section let us go back to the general setting where E is a nuclear space with $(\underline{DN})$ and (Ω) and ϵ the associated exponent sequence, we know

$$\Delta(\Lambda_1(\epsilon)) = \Lambda_1(\epsilon) \subset \Delta(E) \subset \Lambda_\infty(\epsilon)' = \Delta(\Lambda_\infty(\epsilon))$$

and these inclusions give the best fit. We usually have to assume that $\Lambda_1(\epsilon)$ is also nuclear, whereas nuclearity of $\Lambda_\infty(\epsilon)$ follows from the assumption that E is nuclear. We assume ϵ is also stable. Our Theorem 7.4. gives us a sufficient condition in terms of the existence of a local quotient from E into $\Lambda_1(\epsilon)$ for the existence of a complemented subspace isomorphic to $\Lambda_1(\epsilon)$. Our generic theorem (Theorem 6.2.) or Theorem 8.2. gives us a sufficient condition for the existence of a complemented subspace isomorphic to $\Lambda_\infty(\epsilon)$. It may happen that E is isomorphic to $\Lambda_1(\epsilon) \oplus F$ and to $\Lambda_\infty(\epsilon) \oplus G$ for some Fréchet spaces F and G. Since every continuous linear map from $\Lambda_1(\epsilon)$ into $\Lambda_\infty(\epsilon)$ is compact, from the main theorem in [13], we obtain that $\Lambda_1(\epsilon) \simeq E_1 \oplus \mathbb{C}^k$ and $G \simeq G_1 \oplus E_1$ for some Fréchet spaces E_1. Since E_1 is isomorphic to some $\Lambda_1(\alpha)$ by the theorem of Mitiagin and Henkin (Theorem 4.4.) and ϵ is stable, it follows easily that E_1 is isomorphic to $\Lambda_1(\epsilon)$ itself. Hence in this case we have that E is isomorphic to $\Lambda_1(\epsilon) \oplus \Lambda_\infty(\epsilon) \oplus G_1$ for some Fréchet space G_1.

If E has the stronger dominating norm property (DN), we have E is isomorphic to $\Lambda_\infty(\epsilon)$. If E has $(\overline{\Omega})$, which is stronger than (Ω), we know that E in this case is isomorphic to $\Lambda_1(\epsilon)$.

In a rather comprehensive work [45], Vogt generalized most of these results to the setting where $\Lambda_\infty(\alpha)$ is a Schwartz space and E a Fréchet space whose topology is generated by a sequence of semi-inner products. He has used some intricate arguments about operators on Hilbert space to obtain new interpolation theorems and applied these to extend the structure theory. However he has been unable to extend Theorem 5.1. to this setting. If this can be done, then [45] will be shortened considerably. So we have the following open problem.

Question. Let $\Lambda_\infty^{(2)}(\alpha)$ be a Schwartz space and α is stable. Let E be a Fréchet-Hilbert space. If $\Delta(\Lambda_\infty^2(\alpha)) \subset \Delta(E)$, does it follow that E is isomorphic to a subspace of $\Lambda_\infty^{(2)}(\alpha)^N$?

Finally, we note that invariants $(DN), (\underline{DN}), (\Omega)$ and $(\overline{\Omega})$ and other similar invariants used by Vogt and others have been generalized to locally convex spaces by Terzioğlu, Yurdakul and Zahariuta [40].

9. Local Quotient Maps

Let $\lambda(A)$ be a nuclear G_∞-space. If E is a locally convex space such that there is local imbedding of $\lambda(A)$ into E, it is not difficult to prove $\Delta(E) \subset \Delta(\lambda(A))$. Conversely, if E is a nuclear Fréchet space with $(\underline{DN})$ and (Ω) and $\Delta(E) \subset \Delta(\lambda(A)) = \lambda(A)'$ then there is a local imbedding of $\lambda(A)$ into E (Lemma 8.1). In particular, there is a local imbedding of $\Lambda_\infty(\epsilon)$ into E, where ϵ is the associated exponent sequence of E. This enabled us to utilize our generic theorem, Theorem 6.2., in different settings, as we have seen in section 8. In section 7, we have dualized our generic theorem by using the duality between G_∞ and G_1-spaces and local quotients maps, which are related to local imbeddings by taking transposes. Although it is not transparent, this duality was used in the proof of Theorem 8.8. We want to go back now to

Theorem 7.4. We want to seek a relation between diametral dimensions and existence of local quotient maps, similar to Lemma 8.1. in the case of local imbeddings.

Let $\Lambda_1(\alpha)$ be nuclear and set

$$\begin{aligned} B &= \left\{(\xi_n) : \left(\sum |\xi_n|^2\right)^{\frac{1}{2}} \leq 1\right\}, \\ V_k &= \left\{(\xi_n) : \left(\sum |\xi_n|^2 \, q_k^{2\alpha_n}\right)^{1/2} \leq 1\right\} \end{aligned}$$

where $0 < q_k \leq w_{k+1} < 1,\ \lim q_k = 1$. It is easy to get

$$d_n(B, V_k) = q_k^{\alpha_n}.$$

Assume E is a nuclear Fréchet space and suppose there is an absolutely convex, Hilbertian, closed, total and bounded subset A of E such that $\forall k\ \ \exists \theta(k)$ and $C_k > 0$ with

$$(*) \qquad q_k^{\alpha_n} \leq C_k \; d_n(A, U_{\theta(k)})$$

where $U_1 \supset U_2 \supset \cdots$ is a base of closed, absolutely convex neighborhoods of E such that the gauge of each U_k is given by a semi-inner product. Since $\Delta(\Lambda_1) = \Delta_B(\Lambda_1)$, we note that (*) in fact implies $\Delta(E) \subset \Delta(\Lambda_1(\alpha))$.

Let E_k be the Hilbert space associated to the semi-norm of U_k and $E[A] = sp\{A\}$ with the norm $\|x\|_A = \inf\{\rho > 0 : x \in \rho A\}$. $\|\cdot\|_A$ is given by an inner product and so $E[A]$ is a Hilbert space. Let $i_k : E[A] \to E_k$ be the canonical imbedding. By the spectral decomposition theorem, for each k, there is an orthonormal basis (x_n^k) of $E[A]$ and an orthonormal sequence (y_n^k) in E_k such that

$$i_k(x) = \sum_{n=0}^{\infty} d_n(A, U_k)(x|x_n^k)_A y_n^k.$$

Now let us order the set $\{x_n^k : k, n \in N\}$ into a sequence by using the bijection $\beta : N \to N \times N$ defined by $\beta^{-1}(k, n) = 2^{k-1}(2n-1)$ and apply the Gram-Schmidt process. Here we are inspired by the proof of the imbedding theorem, Theorem 5.1., by Ramanujan and Terzioğlu [33]. We obtain this way a new orthonormal basis (x_n) of $E[A]$. If $2^k n < m$ we have $(x_m|x_n^k)_A = 0$. So if m_k is the smallest integer greater than or equal to $m/2^k$, we have

$$x_m = \sum_{n=m_k} (x_m|x_n^k)_A \; x_n^k.$$

Using nuclearity of $\Lambda_1(\alpha)$ for k we find k' and $\rho_k > 0$ with

$$\left(\sum |\xi_n|^2 q_k^{2\alpha_n}\right)^{1/2} \leq \rho_k \quad \sup |\xi_n| q_{k'}^{\alpha_n}, \quad (\xi_n) \in \Lambda_1(\alpha).$$

Let $k" = \theta(k')$. So by (*) we have some $C_{k'} > 0$ with

$$q_{k'}^{\alpha_n} \leq C_k, \;\; d_n(A, U_{k"}), \quad n \in N.$$

For $x \in E[A]$, we have

$$\begin{aligned} |(x|x_m)_A| &\leq \sum_{n=m_{k''}} |(x_m|x_n^{k''})_A|\,|(x|x_n^{k''})_A| \\ &\leq \left(\sum_{n=m_{k''}} \frac{|(x_m|x_n^{k''})_A|^2}{d_n(A,U_{k''})^2}\right)^{1/2} \cdot \left(\sum_{n\in N} |(x|x_n^{k''})_A|^2\, d_n(A,U_{k''})^2\right)^{1/2}. \end{aligned}$$

Remembering $m \geq m_k$ from above we get

$$|(x|x_m)|q_{k'}^{\alpha_m} \leq C_{k'}\,\|x\|_{k''}$$

and therefore

$$\left(\sum |(x|x_m)_A|^2\, q_k^{2\alpha_m}\right)^{1/2} \leq \rho_k C_{k'}\,\|x\|_{k''}, \quad x \in E[A].$$

So if we define $T_0 : sp\{A\} \to \Lambda_1(\alpha)$ by setting $T_0(x) = ((x|x_m)_A)$ we have that T_0 is a continuous linear map of the subspace $sp\{A\}$ of E into $\Lambda_1(\alpha)$. Let us extend this by continuity to $T : E \to \Lambda_1(\alpha)$, remembering that A is total in E.

Let $\xi \in B$ and $\xi_i = (\xi_1, \ldots, \xi_i, 0, \ldots, 0\ldots)$. Let

$$y_i = \sum_{m=1}^{i} \xi_m x_m.$$

We have $T(y_i) = T_0(y_i) = \xi_i$ and $y_i \in A$ for each i. Since $\lim \xi^i = \xi$ in $\Lambda_1(\alpha)$, we have that $B \subset \overline{T(A)}$. Hence under the assumption (*) we have proved that there is a local quotient map $T : E \to \Lambda_1(\alpha)$.

We now seek a relation between diametral dimension and the technical condition (*). For this purpose we go back to the two different versions of the diametral dimension. (cf. [8]). We know $\Delta(E) \subset \Delta_B(E)$. If E is a Montel space, then $\lim d_n(A,U) = 0$ for every $A \in \mathcal{B}(E)$ and $U \in (E)$. E is a Schwartz space if and only if for every $U \in \mathcal{U}(E)$ there is a $V \in \mathcal{U}(E)$ with $\lim d_n(V,U) = 0$. On the other hand we know that there is a Köthe space $\lambda(A)$ which is a Montel space but $\lambda(A)$ is not a Schwartz space [20]. For this space we have $\Delta(\Lambda(A)) = c_0$ but $\ell_\infty \subset \Delta_B(\lambda(A))$.

Proposition 9.1. *For every Fréchet-Schwartz space E we have* $\Delta(E) = \Delta_B(E)$.

Proof. We can assume without loss of generality that E has a base of absolutely convex, closed neighborhoods $U_1 \supset U_2 \supset \cdots$ such that $\lim_{n\to\infty} d_n(U_{k+1}, U_k) = 0$ for every k. Let us assume $(\xi_n) \in \Delta_B(E)$ but $(\xi_n) \notin \Delta(E)$. This means that there is a k so that for every $m > k$ the sequence $(\xi_n d_n(U_m, U_k))$ does not converge to zero, or

$$\limsup_n \xi_n d_n(U_m, U_k) > 0.$$

By a diagonal process and without any loss of generality we assume

$$\lim_{n\to\infty} \xi_{i_n} d_{i_n}(U_m, U_k) = \ell(m) > 0.$$

To simplify notation and without any loss of generality we assume

$$\lim_{n\to\infty} \xi_n d_n(U_m, U_k) = \ell(m).$$

Since $0 \leq d_n(U_{m+1}, U_k) \leq d_n(U_m, U_k)$, we have that

$$a_n = \lim_{m\to\infty} \xi_n d_n(U_m, U_k)$$

exists for each n. For each n there is some $m_0 = m_0(n)$ such that for all $m \geq m_0$ we have

$$0 \leq \xi_n d_n(U_m, U_k) - a_n < 1/n.$$

Thus we can find $k < k(1) \leq k(2) \leq \cdots$ such that $\lim k(n) = \infty$ and

$$\lim_{n\to\infty} (\xi_n d_n(U_{k(n)}, U_k) - a_n) = 0.$$

Since $\lim_{n\to\infty} d_n(U_m, U_k) = 0$ for each $m > k$, for every n we can find a subset $A_n \subset U_{k(n)}$ such that A_n is the absolutely convex hull of n-points and $d_n(A_n, U_k) = d_n(U_{k(n)}, U_k)$. Let A be the absolutely convex and closed hull of $\cup A_n$. A is bounded and further

$$d_n(U_{k(n)}, U_k) = d_n(A_n, U_k) \leq d_n(A, U_k).$$

So $\lim \xi_n d_n(U_{k(n)}, U_k) = 0$ and therefore $\lim a_n = 0$. On the other hand for each m there is some $n_0 = n_0(m)$ such that

$$0 \leq \xi_n d_n(U_m, U_k) - a_n < \frac{\ell(m)}{2}$$

for all $n \geq n_0$. This implies $\ell(m) < (\ell(m)/2)$. Hence $\ell(m) = 0$ for some m. □

We can now relate condition (*) to diametral dimension.

Proposition 9.2. *Let E be a nuclear F-space. Then (*) holds if and only if $\Delta(E) \subset \Lambda_1(\alpha)$.*

Proof. We have $\Delta(E) = \Delta_B(E) \subset \Lambda_1(\alpha)$ by Proposition 9.1. Given k there is some $A_k \in \mathcal{B}(E)$ and $\theta(k)$ such that

$$\sup_n |\xi_n| q_k^{\alpha_n} \leq \rho(k) \sup |\xi_n| d_n(A_k, U_{\theta(k)}).$$

Here we use the continuity of the inclusion $\Delta_B(E) \subset \Lambda_1(\alpha)$. This means

$$q_k^{\alpha_n} \leq \rho(k) d_n(A_k, U_{\theta(k)}).$$

Now we find $\sigma(k)$ with $\sigma(k)A_k \subset U_k$. So $\cup\sigma(k)A_k$ is a bounded subset. Let A be the absolutely convex, closed hull of $\cup\sigma(k)A_k$. Then there is some $c_k > 0$ such that

$$q_k^{\alpha_n} \leq c_k d_n(A, U_{\theta(k)})$$

for all $n \in N$. The converse is trivial. □

We have therefore proved the following result.

Proposition 9.3. *There is a local quotient from the nuclear Fréchet space E into $\Lambda_1(\alpha)$ if and only if $\Delta(E) \subset \Delta(\Lambda_1(\alpha)) = \Lambda_1(\alpha)$.*

We apply now this result to Theorem 7.4. to obtain the main result of this section which should be compared to Theorem 8.2. Theorem 9.4.

Aytuna has recently proved that $\Delta(\mathcal{O}(M))$ is either equal to $\Lambda_1(n^{1/d})$ or to $\Lambda_\infty(n^{1/d})'$. This remarkable result together with Theorem 9.4. gives us the following Corollary 9.5.

Theorem 9.4. *Let E be a nuclear Fréchet space with properties $(\underline{DN})$ and (Ω). Assume $\Delta(E) = \Delta(\Lambda_1(\alpha))$, where α is stable. Then E has a complemented subspace which is isomorphic to $\Lambda_1(\alpha)$.*

Aytuna has recently proved that $\Delta(\mathcal{O}(M))$ is either equal to $\Lambda_1(n^{1/d})$ or to $\Lambda_\infty(n^{1/d})'$. This remarkable result together with Theorem 9.4. gives us the following:

Corollary 9.5. *Let M be a d-dimensional Stein manifold. Then either $O(M) \simeq O(\Delta^d) \oplus F_1$ or $O(M) \simeq O(\mathbb{C}^d) \oplus F_2$ for some F_1 or F_2.*

Of course there are cases where both $O(\mathbb{C}^d)$ and $O(\Delta^d)$ are complemented subspaces of $O(M)$. From $O(\Delta^d) \oplus F_1 \simeq O(\mathbb{C}^d) \oplus F_2$, remembering that every continuous linear map from $O(\Delta^d)$ into $O(\mathbb{C}^d)$ is compact, we use the main theorem in [13] to get $O(\Delta^d) \simeq E_1 \oplus \mathbb{C}^k$ for some k and $F_2 \simeq E_1 \oplus G$ for some Fréchet space G. From the Mitiagin-Henkin theorem and stability of $(n^{1/d})$ we have $E_1 \simeq O(\Delta^d)$. Hence

$$O(M) \simeq O(\Delta^d) \oplus O(\mathbb{C}^d) \oplus G.$$

Let us go back to the beginning of this section, where we have assumed E is a nuclear Fréchet space with $\Delta(E) \subset \Lambda_1(\alpha)$ and showed that there is an absolutely convex, closed, total, Hilbertian subset A such that for each k there is $\theta(k)$ and $C_k > 0$ with

$$q_k^{\alpha_n} \leq c_k d_n(A, U_{\theta(k)}).$$

We have constructed an orthonormal basis (x_m) of $E[A]$ such that the map $T : E \to \Lambda_1(\alpha)$ defined by $T(x_m) = e_m$ is a local quotient. Let us continue along this line of thinking to get an improved version of Theorem 9.4. much more directly.

We simply assume $\Delta(E) = \Lambda_1(\alpha)$ where α is stable. Now $\Lambda_1(\alpha) \subset \Delta_B(E)$ gives us that $\xi \in \Lambda_1(\alpha) \Rightarrow \lim \xi_n d_n(A, U_k) = 0$. Let us denote by Δ_k the set of all sequences (ξ_n) with $\lim \xi_n d_n(A, U_k) = 0$. Δ_k is a Banach space with norm

$$\sup |\xi_n| \, d_n(A, U_k)$$

and $\Delta_k \subset \Delta_{k+1}$ for each k. So

$$\Lambda_1(\alpha) \subset \cup_{k=1}^\infty \Delta_k.$$

By the Grothendieck factorization theorem we have

$$\Lambda_1(\alpha) \subset \Delta_{k_0}$$

for some k_0. This means that for every $k \geq k_0$ and for every j we have $D(k_{ij}) > 0$ such that

$$d_n(A, U_k) \leq D(k, j) q_j^{\alpha_n}$$

holds. We have

$$x_i = \sum_{n=i_k} (x_i|x_n^k)x_n^k$$

where i_k is the smallest integer $\geq i/2^k$. So

$$\|x_i\|_k \leq d_{i_k}(A, U_k) \leq D\ q_k^{\alpha_{i_k}}.$$

By stability of $\Lambda_1(\alpha)$ we choose now k' with $q_k^{\alpha_{i_k}} \leq q_{k'}^{\alpha_i}$. Hence

$$\|x_i\|_k \leq D\ q_{k'}^{\alpha_i}.$$

Hence, if we define $S : \Lambda_1(\alpha) \to E$ by $S(e_i) = x_i$, we see that S is continuous. Note that TS is equal to the identity I of $\Lambda_1(\alpha)$ and so $ST : E \to E$ is a continuous projection. For any $\xi \in \Lambda_1(\alpha)$ we have $\xi = TS(\xi)$ and so $T(E) = \Lambda_1(\alpha)$. $S : \Lambda_1 \to E$ is certainly one to one and if $\lim S(\xi_j) = y$ for some sequence (ξ_j) in $\Lambda_1(\alpha)$, we have $\lim \xi_j = \lim TS(\xi_j) = T(y)$. So $ST(y) = y$ and therefore the range of S is closed, which means that S maps $\Lambda_1(\alpha)$ isomorphically in E. Hence the range of the projection $ST : E \to E$ is isomorphic to $\Lambda_1(\alpha)$. We state what we have proved as our final result.

Theorem 9.6. *Let E be a nuclear Fréchet space with $\Delta(E) = \Lambda_1(\alpha)$ where $\Lambda_1(\alpha)$ is stable. Then E has a complemented subspace which is isomorphic to $\Lambda_1(\alpha)$.*

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